Chapter 12. The continuous and discrete Fourier transforms

Approximating a signal with the Fourier series results in a collection of simple cosine and sine waves with varying frequency, amplitude, and phase. Simply put, if we know amplitudes, frequencies, and phase amounts, then all we need is time. These four parameters fully define a complex signal. We are used to thinking of signals as a function of time, but, in an existing signal, any of the four parameters are functions of the remaining ones. For example, if we know time and amplitudes, all we need is frequencies and phase.

12.1. Definitions

The Fourier transform translates something that is a function of time given specific amplitudes, frequencies, and phase amounts into something that is a function of frequencies and phase amounts for specific amplitudes and time. As any other transform, the Fourier transform translates one type of function into another. We cannot derive the Fourier transform as it is not something that is derived. It is simply something that is defined.

The continuous Fourier transform of the function $x(t)$ is defined as follows.

$$H(s) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j\, t\, s} dt \tag{12.1}$$

The inverse continuous Fourier transform is

$$x(t) = \int_{-\infty}^{\infty} H(s) e^{2\pi j\, t\, s} ds \tag{12.2}$$

The discrete Fourier transform (DFT) of the function $x(k)$ is defined as follows.

$$H(n) = \sum_{k=0}^{N-1} x(k) e^{-\frac{2\pi\, j\, k\, n}{N}} \tag{12.3}$$

$n = 0, 1, \ldots, N - 1.$

The inverse discrete Fourier transform (inverse DFT) is

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} H(n) e^{\frac{2\pi j k n}{N}} \tag{12.4}$$

$k = 0, 1, \ldots, N-1$.

Euler's formula[8] allows us to write, for example, the discrete Fourier transform also as follows.

$$H(n) = \sum_{k=0}^{N-1} x(k)\left(cos\left(\frac{2\pi\, k\, n}{N}\right) - j\, sin\left(\frac{2\pi\, k\, n}{N}\right)\right) \tag{12.5}$$

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} H(n)\left(cos\left(\frac{2\pi\, k\, n}{N}\right) + j\, sin\left(\frac{2\pi\, k\, n}{N}\right)\right) \tag{12.6}$$

We have already used formulae similar to the ones above. The frequency content computation of chapter 5 is simply the forward Fourier transform. In chapter 7, we derived the general form of a low pass filter by taking a desired magnitude response, which is a function of the frequency, summing up all cosine functions that fit in that desired magnitude response, and scaling them by $(2 / f_s)$. The filter $a(k)$ was computed as the sum of integer frequencies m up to f_c, which is simply a form of the inverse Fourier transform. We will confirm these findings after a simple DFT example.

12.2. A simple DFT example

Take a simple sine wave and sample it over N points for a period of 1 second. Suppose that the simple wave has the integer frequency f, amplitude A, and phase m. The values of this simple wave would be

$$\begin{aligned} &A\, sin\left(\frac{2\pi\, f\, (k-m)}{N}\right) \\ &\quad = A\, cos\left(\frac{2\pi\, f\, m}{N}\right) sin\left(\frac{2\pi\, f\, k}{N}\right) - A\, sin\left(\frac{2\pi\, f\, m}{N}\right) cos\left(\frac{2\pi\, f\, k}{N}\right) \\ &\quad = B\, cos\left(\frac{2\pi\, f\, k}{N}\right) + C\, sin\left(\frac{2\pi\, f\, k}{N}\right) \end{aligned} \tag{12.7}$$

where $B = -A\ \sin(2\pi f\, m\ /\ N)$ and $C = A\ \cos(2\pi f\, m\ /\ N)$. Using equation 12.5, we can compute the DFT as follows.

[8] Euler's formula states that $e^{jx} = \cos(x) + j\sin(x)$.

$$H(n) = \sum_{k=0}^{N-1} \left(B\, cos\left(\frac{2\pi\, f\, k}{N}\right) + C\, sin\left(\frac{2\pi\, f\, k}{N}\right)\right)\left(cos\left(\frac{2\pi\, n\, k}{N}\right) - j\, sin\left(\frac{2\pi\, n\, k}{N}\right)\right) \qquad (12.8)$$

for n = 0, 1, …, N – 1. Here we note that most of these functions will be orthogonal over the sampling period, except in *two* cases – if $n = f$ or $N - n = f$. In the second case, we note that

$$\begin{gathered} cos\left(\frac{2\pi\, k\, (N-n)}{N}\right) = cos\left(2\pi\, k - \frac{2\pi\, k\, n}{N}\right) = cos\left(\frac{2\pi\, k\, n}{N}\right) \\ sin\left(\frac{2\pi\, k\, (N-n)}{N}\right) = sin\left(2\pi\, k - \frac{2\pi\, k\, n}{N}\right) = -\, sin\left(\frac{2\pi\, k\, n}{N}\right) \end{gathered} \qquad (12.9)$$

The transform translates to

$$H(n) \approx \begin{cases} B \sum_{k=0}^{N-1} cos^2\left(\frac{2\pi\, k\, n}{N}\right) - j\, C \sum_{k=0}^{N-1} sin^2\left(\frac{2\pi\, k\, n}{N}\right) \\ \qquad\qquad ,\quad n = f, N - f \\ \qquad 0, \qquad n \neq f, N - f \end{cases} \qquad (12.10)$$

From the computations of RMS amplitude in chapter 4 we know that the two sums for a large enough N evaluate to N / 2 and so the transform is simply

$$H(n) \;\approx \begin{cases} \frac{N}{2}(B - j\, C), & n = f, N - f \\ \quad 0, & n \neq f, N - f \end{cases} \qquad (12.11)$$

Thus, the DFT of a simple wave with frequency f produces a function $H(n)$ that indicates something about the single wave at components $n = f$ and $n = N - f$ and is zero for other values of n. Specifically, at $n = f$ and $n = N - f$

$$|H(n)| \approx \sqrt{Re(H(n))^2 + Im(H(n))^2} = \frac{N}{2}\sqrt{B^2 + C^2} = \frac{N}{2}\, A \qquad (12.12)$$

Similarly

$$atan2\big(Im(H(n)), Re(H(n))\big) = atan2(N\, C, N\, B) = -\frac{2\pi\, f\, m}{N} \qquad (12.13)$$

The discrete Fourier transform $H(n)$ of a real valued signal $x(k)$ shows the amplitude of the simple wave with frequency n in the real valued signal $x(k)$ (or, more appropriately as we will see below, the nth simple wave component in the real valued signal $x(k)$)

$$A_n = 2\frac{|H(n)|}{N} = 2\frac{\sqrt{Re(H(n))^2 + Im(H(n))^2}}{N} \tag{12.14}$$

and the phase of the simple wave with frequency n in $x(k)$

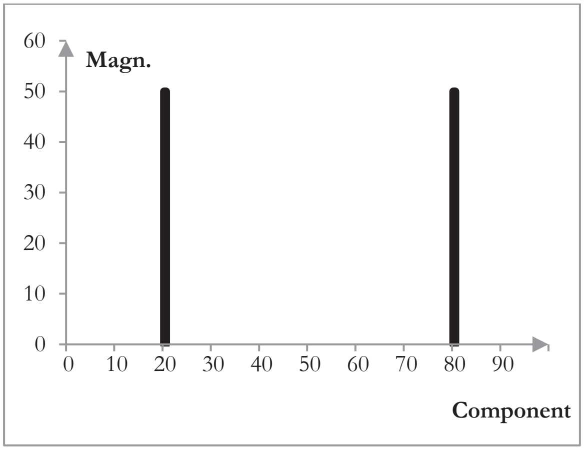
$$\Phi_n = atan2\big(Im(H(n)), Re(H(n))\big) \tag{12.15}$$

This result is no different than the result in equations 5.15 through 5.18, which we introduced during our discussion of the Fourier series. The Fourier transform is a natural extension of the Fourier series. There are three motivations for using the Fourier transform rather than the Fourier series. First, the Fourier transform allows for shorter notation. Second, the use of complex numbers may come as a surprise, but it allows the Fourier transform to return a complex number, which means two separate values that can be used to describe the amplitude and the phase of the signal separately. Third, the Fourier transform has an inverse transform, which is easy to describe and which has its own uses and we will see below.

Suppose that we compute, for example, the DFT of a simple sine wave at 10 Hz, using 100 DFT points over 2 seconds of the wave. The magnitude of the resulting DFT components will be as in figure 51. The DFT will be non-zero at components 20 and 80 and (approximately) zero at all other components. Since we are sampling with 100 points over 2 seconds, we have effectively sampled the wave with the sampling frequency 50 Hz. This means that we have used the DFT with components at 0.5 Hz, 1 Hz, 1.5 Hz, …, 49.5 Hz[9]. The 20th component is 10 Hz and it makes sense that we have a non-zero value at this component.

[9] Integer frequencies are orthogonal over 1 second. Multiples of 0.5 Hz are orthogonal over 2 seconds.

Figure 51. DFT magnitude of a simple wave



Magnitude of the discrete Fourier transform of a simple wave at 10 Hz. The transform was computed with 100 points over two seconds.

These results are as expected[10]. We specifically chose to compute the DFT with N uniform points over 2 seconds. How we choose the points of the DFT defines what the DFT components are. In this case, the DFT components represent the frequencies between 0 and N / 2. Most often, the DFT is computed with the first N points of a signal sampled at the sampling frequency f_s. In this case, the DFT components are the frequencies 0, f_s / N, 2 f_s / N, …, (N – 1) f_s / N.

It is also important to note that, in this simple example, the frequency that we examined – 10 Hz – happens to be one of the DFT components. When this is not the case, the resulting figure is not as clean. This is examined further in chapter 14.

According to the discussion above, in addition to getting a nonzero value at component 20, we also get a nonzero value at component 80 = 100 – 20. The magnitude of the DFT at both components is 100 / 2 = 50.

When the input to the DFT or the inverse DFT is real valued and not complex, the DFT produces complex numbers such that $H(N - 1 - n)$ is the complex conjugate of $H(n)$. The information for $n > N / 2$ is redundant.

[10] The DFT then poses one interesting question, namely what the DFT component at 0 represents. The magnitude of the DFT component at 0 for a signal of the form $x(k) = A_0 + A_1 \sin(2\pi\, k\, f\, /\, N)$ would be $N\, A_0$. Thus, although the DFT component at 0 is usually meaningless for digital signal processing, in analog signals it represents DC current.

The fact that the DFT of a real valued signal produces repeated and redundant information is uncomfortable. We have two choices. First, we can ignore the second half DFT. Second, we can feed the DFT with complex numbers. The complex frequency 10 Hz sampled over *N*, for example, is

$$e^{-\frac{2\pi\, j\, 10\, k}{N}} \tag{12.16}$$

When the DFT is used on complex data, however, the results are different. Take the simple wave

$$e^{-\frac{2\pi\, j\, f\, k}{N}} = cos\left(\frac{2\pi\ f\ k}{N}\right) - j\, sin\left(\frac{2\pi\ f\ k}{N}\right) \tag{12.17}$$

The component *H*(*n*) of the DFT at *n* = *f* would be

$$\begin{aligned} H(n) \approx &\sum\nolimits_{k=0}^{N-1}\left(cos^2\left(\frac{2\pi\ k\ n}{N}\right) + sin^2\left(\frac{2\pi\ k\ n}{N}\right)\right) \\ &- j\sum\nolimits_{k=0}^{N-1}\left(sin\left(\frac{2\pi\ k\ n}{N}\right)cos\left(\frac{2\pi\ k\ n}{N}\right) + sin\left(\frac{2\pi\ k\ n}{N}\right)cos\left(\frac{2\pi\ k\ n}{N}\right)\right) = N \end{aligned} \tag{12.18}$$

The difference is that this result is twice as large as the result of the DFT of a real valued signal.

The discrete Fourier transform *H*(*k*) of a complex valued signal *x*(*k*) shows the amplitude of the n^th^ simple wave component in the complex valued signal *x*(*k*)

$$A_n = \frac{|H(n)|}{N} = \frac{\sqrt{Re(H(n))^2 + Im(H(n))^2}}{N} \tag{12.19}$$

and the phase of the n^th^ simple wave component in *x*(*k*)

$$\Phi_n = arg(H(n)) = atan2\big(Im(H(n)), Re(H(n))\big) \tag{12.20}$$

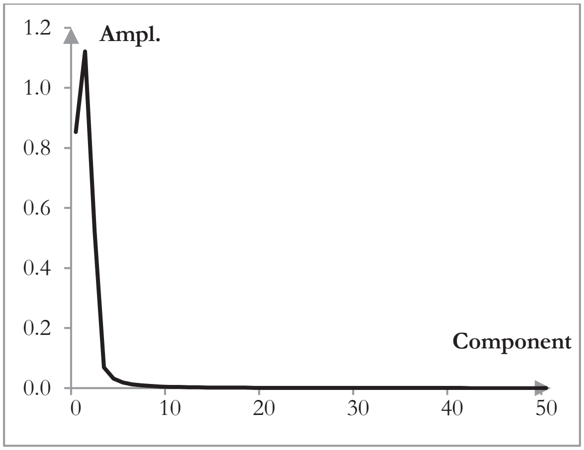
12.3. The DFT and FIR filters

Since the low pass filter of chapter 7 is simply a complex signal composes of the simple waves with frequency less than the cutoff frequency f_c, we should be able to use the DFT to compute the frequency (magnitude and phase) response of the low pass filter. As above, we note that this low pass filter is real valued and hence we can dispose of half of the DFT components. We also note that we have already scaled the low pass filter of length *N* by 2 / *N*. Thus, we can compute the standard DFT over *N* points, but to compute the magnitude, we will not scale by 2 / *N*. The resulting computation for the magnitude response of the filter is equivalent to the one we already used in equation 8.5.

$$|H(n)| = \sqrt{\left(\sum_{k=0}^{N-1} a(k)\, sin\left(\frac{2\pi\ k\ n}{N}\right)\right)^2 + \left(\sum_{k=0}^{N-1} a(k)\, cos\left(\frac{2\pi\ k\ n}{N}\right)\right)^2} \qquad (12.21)$$

The fact that the DFT is computed with a small set of components – the length of the filter – means that the magnitude response would be sparse in the frequency domain, as shown on figure 52. The magnitude response in figure 52 employs a DFT of *N* = 101 points, which is the same as the length of the filter. The cutoff frequency is between components 2 and 3, which, given the sampling frequency of f_s = 2000 Hz, translates to between 39.6 Hz and 59.4 Hz (sampling 101 times over 2000 Hz implies 19.8 Hz per component). Note that figure 52 is limited to component 50, or half of *N*.

Figure 52. Magnitude response with the DFT



The magnitude response of this low pass filter of *N* = 101 points was computed with DFT of *N* = 101 components. The cutoff frequency is between components 2 and 3.

To compute the phase response of our low pass filter, we note that the filter has coefficients that are symmetric around the middle, i.e., $a(k) = a(N-1-k)$. Assume for a moment, that Re($H(n)$) > 0. Then

$$\begin{aligned} \Phi_n &= atan2\left(Im\left(H(n)\right), Re\left(H(n)\right)\right) = atan\left(\frac{Im\left(H(n)\right)}{Re\left(H(n)\right)}\right) \\ &= atan\left(\frac{\sum_{k=0}^{N-1} a(k)\, sin\left(\frac{2\pi\, k\, n}{N}\right)}{\sum_{k=0}^{N-1} a(k)\, cos\left(\frac{2\pi\, k\, n}{N}\right)}\right) \end{aligned} \tag{12.22}$$

Since $a(k) = a(N - 1 - k)$,

$$\begin{aligned} a(k)\, sin\left(\frac{2\pi\, k\, n}{N}\right) + a(N-1-k)\, sin\left(\frac{2\pi\, (N-1-k)n}{N}\right) \\ = 2\, a(k)\, sin\left(\frac{2\pi\, (N-1)n}{2\, N}\right) cos\left(\frac{2\pi\, (N-1-2\, k)n}{2\, N}\right) \end{aligned} \tag{12.23}$$

Similarly,

$$\begin{aligned} a(k)\, cos\left(\frac{2\pi\, k\, n}{N}\right) + a(N-1-k)\, cos\left(\frac{2\pi\, (N-1-k)n}{N}\right) \\ = 2\, a(k)\, cos\left(\frac{2\pi\, (N-1)n}{2\, N}\right) cos\left(\frac{2\pi\, (N-1-2\, k)n}{2\, N}\right) \end{aligned} \tag{12.24}$$

The phase response, assuming the filter is of odd length, is as follows.

$$\Phi_n = atan\left(\frac{2\, sin\left(\frac{2\pi\, n}{N}\frac{N-1}{2}\right)}{2\, cos\left(\frac{2\pi\, n}{N}\frac{N-1}{2}\right)} \frac{\left(1 + \sum_{k=0}^{\frac{N-1}{2}}\left(a(k)\, cos\left(\frac{2\pi\, n}{N}\, \frac{N-1-2k}{2}\right)\right)\right)}{\left(1 + \sum_{k=0}^{\frac{N-1}{2}}\left(a(k)\, cos\left(\frac{2\pi\, n}{N}\, \frac{N-1-2k}{2}\right)\right)\right)}\right) \tag{12.25}$$

The "1 +" in the numerator and denominator handle the case when the filter is of odd length *N*. If the filter has even number of points *N*, the computations would be the same but without "1 +". Thus,

$$\Phi_n = atan\left(\frac{sin\left(\frac{2\pi\, n}{N}\, \frac{N-1}{2}\right)}{cos\left(\frac{2\pi\, n}{N}\, \frac{N-1}{2}\right)}\right) = \frac{2\pi\, n}{N}\, \frac{N-1}{2} \tag{12.26}$$

As discussed in chapter 8, the phase response is linear with respect to the frequency *n* and produces a delay of $(N-1)\ /\ 2$ samples. These computations assume that $\mathrm{Re}(H(n)) > 0$, which allows us to replace atan2 with atan, but the computations are similar in other cases.

Fact: The linear phase response of symmetric FIR filters is used in radars. Radars determine distance by measuring the phase of several frequencies in the frequency spectrum. If the radar should use a FIR filter, it is helpful if the FIR filter introduces minimal or at least well-known distortion of the phase.

12.4. Using the inverse DFT

If the DFT produces the magnitude response of a filter, then we should be able to produce a filter by taking the inverse DFT from a desired magnitude response. In chapter 7, we created a low pass filter by summing up the simple waves up to the cutoff frequency.

$$a(k) = \frac{2}{f_s}\sum_{m=1}^{f_c} cos\left(\frac{2\pi\ m\ k}{f_s}\right) \tag{12.27}$$

$k = 0, 1, \ldots, f_s - 1$. This equation can also be written as

$$a(k) = 2\left(\frac{1}{f_s}\sum_{m=1}^{\frac{f_s}{2}} B_m\, cos\left(\frac{2\pi\ m\ k}{f_s}\right)\right), B_m = \begin{cases}1, & m \le f_c\\ 0, & m > f_c\end{cases} \tag{12.28}$$

where B_m is the desired magnitude response of the filter, equal to 1 up to the cutoff frequency f_c and zero afterwards. The upper limit of the sum is $f_s\ /\ 2$ as suggested by the Nyquist-Shannon sampling theorem.

This formula is very similar to the inverse DFT, except that it only takes the real part of the inverse DFT, hence only the cosines, is limited to half of the inverse DFT, hence the sum goes up to only $f_s\ /\ 2$, and is scaled by 2. Thus, this formula samples the frequency content and uses computations similar to the inverse DFT to create a filter. We are not limited to sampling only at f_s, and we can sample at any evenly spaced *N* points. In other words, the low pass filter can be created by

$$a(k) = 2\left(\frac{1}{N}\sum_{n=0}^{N/2} B_n\, cos\left(\frac{2\pi\ n\ k}{N}\right)\right), B_n = \begin{cases}1, & \dfrac{n\,f_s}{N} \le f_c\\ 0, & \dfrac{n\,f_s}{N} > f_c\end{cases} \tag{12.29}$$

$k = 0, 1, \ldots, N - 1$. We note that cosine functions are symmetric around the origin and that the filter $a(k)$ could similarly be symmetric around the origin, if we are after a linear phase response.

We can then extend the sum and remove the scaling factor of 2, for k = 0, 1, …, N – 1 as follows.

$$a(k) = \frac{1}{N}\sum_{n=-N/2}^{N/2} B_n \cos\left(\frac{2\pi\, n\, k}{N}\right), B_n = \begin{cases} 1, & f_c \le \frac{n\, f_s}{N} \le f_c \\ 0, & \left|\frac{n\, f_s}{N}\right| > f_c \end{cases} \tag{12.30}$$

Note the natural addition of n = 0 to the sum. Since B_n represents the desired filter magnitude response, the magnitude response should remain at 1 as the frequency approaches zero.

Finally, to make the filter symmetric around (N – 1) / 2, for the same B_n and k, we shift the filter as follows.

$$a(k) = \frac{1}{N}\sum_{n=-N/2}^{N/2} B_n \cos\left(\frac{2\pi\, n\, \left(k - \frac{N-1}{2}\right)}{N}\right) \tag{12.31}$$

This filter formula resembles the inverse DFT, but it is *not* the inverse DFT. We *do not use the inverse DFT to produce a FIR filter*[11]. There are important differences between the inverse DFT and the filter computation above. First, the formula above uses only the real part of the inverse DFT. Second, the formula is shifted both in n, as the sum runs from –N / 2 to N / 2, and in k, as the sum uses k – (N – 1) / 2.

The formula above is the real portion of a *generalized or shifted form of the inverse complex valued DFT*, which is usually written as

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} H(n) e^{\frac{2\pi\, j\, (k+a)\, (n+b)}{N}} \tag{12.32}$$

Note that we could easily add the imaginary part of the inverse DFT to equation 12.31 above. Since sine functions are odd (i.e., sin(-x) = -sin(x)), the sum of the sine values for n = -N / 2, …, N / 2 would simply be zero. In other words, by designing a desired magnitude response B_n that is symmetric around the origin, we created an inverse DFT and thus a filter that contains only real valued coefficients.

[11] If we wanted to use the non-generalized form of the DFT as defined in equation 12.3, we would have to scale the results by 2. If we include a value for the magnitude response at frequency zero, that value would be 0.5 and not 1.

Using a desired magnitude response that includes negative frequencies may be troublesome, since negative frequencies do not mean much. The technique of using a symmetric magnitude response to produce a real valued filter is, however, common.

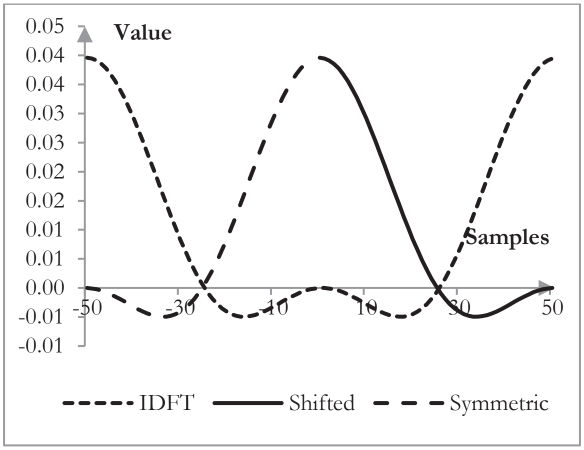
Suppose that, as before, we need a filter is of length $N = 101$ and cutoff frequency $f_c = 40$ Hz, given the sampling frequency $f_s = 2000$ Hz. To use the inverse DFT to create the filter, we must first produce the desired magnitude response $|H(n)|$. We can use the inverse DFT with $N = 101$ points to produce the 101 points of the filter, but we must be careful to define a magnitude response that is symmetric around the origin. We will use $N = 101$ DFT components between $-f_s / 2$ and $f_s / 2$. Each component will be responsible for $2000 / 101 \approx 19.8$ Hz. DFT component 0 is at frequency 0. DFT components -1 and 1 are at -19.8 Hz and 19.8 Hz respectively. DFT components -2 and 2 are at -39.6 Hz and 39.6 Hz respectively, and so on.

An appropriate magnitude response might be $|H(n)|$ with values of 1 at components -2, -1, 0, 1, 2 and zero everywhere else. One of the benefits of using the inverse DFT, however, is that we can include a transition band in the desired magnitude response. If we want to include a transition band, we might define a magnitude response that is 1 at components -1, 0, and 1, that is 0.5 at components -2 and 2, and that is zero everywhere else.[12] This could be an appropriate magnitude response of a low pass filter with cutoff frequency $f_c = 40$ Hz, as we might want a smoother transition band to mitigate the Gibbs phenomenon ripples.

The results of the inverse DFT on $|H(n)|$ are shown on figure 53 (solid and dashed). The magnitude response taken with the forward DFT is displayed in figure 54. It looks great, but it is misleading. Since the filter has a low cutoff frequency and since the DFT has few points, the DFT produces approximately the desired magnitude response. The more detailed magnitude response computation on figure 55 compares the response of this filter to the low pass filter created in chapter 7. Including a transition band in the desired magnitude response $|H(n)|$ produces a filter with a slower transition band, but with smaller ripples and better stop band attenuation.

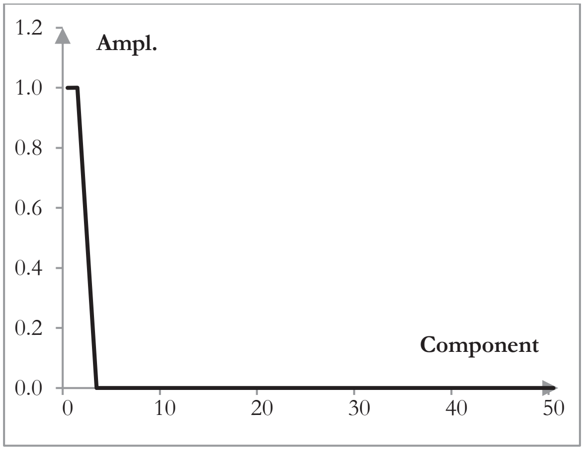
[12] We will use this type of adjustment below to derive the family of Hamming windows.

Figure 53. Shifting and adding to the inverse DFT filter



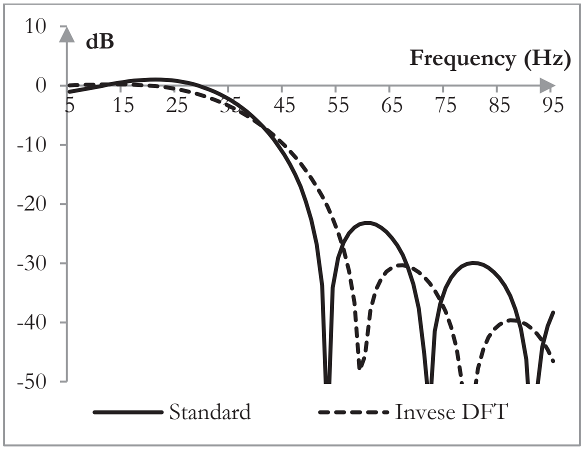
Computing the inverse DFT produces a filter with energy in the sides (dotted). Shifting the filter from k to $k - (N - 1) / 2$ (solid) and adding $k = -(N - 1) / 2, \ldots, -1$ (dashed) produces a symmetric filter with energy in the middle.

Figure 54. Magnitude response of an inverse DFT filter computed with the DFT



At low cutoff frequencies, the magnitude response of the inverse DFT filter computed with the DFT of few points matches the desired frequency response.

Figure 55. A more precise computation of the magnitude response of an inverse DFT filter



Magnitude response of the inverse DFT filter (dashed) and of the standard filter. The desired magnitude response for the inverse DFT allowed for a transition band.

Fact: Many desired magnitude responses can be converted into a filter with the inverse Fourier transform or the inverse discrete Fourier transform to produce, for example, nice parametric equalizers – equalizers in which the user can adjust both the gain and the frequencies at which changes in the gain occur.

12.5. Sampling of the continuous inverse Fourier transform

The continuous-time Fourier transform translates a signal *x*(*t*) that is a function of time for specific amplitudes, frequencies, and phase amounts into a signal *H*(*x*(*t*)) that is a function of frequencies and phase amounts for specific amplitudes and time. The inverse Fourier transform translates functions of frequencies and phase amounts into signals that are functions of time.

Suppose that we use a continuous magnitude response function rather than a discrete desired magnitude response function.

$$|H(f)| = \begin{cases} 1, & -\frac{f_c}{f_s} \leq f \leq \frac{f_c}{f_s} \\ 0, & |f| > \frac{f_c}{f_s} \end{cases} \tag{12.33}$$

This is an appropriate magnitude response, symmetric around the origin, of some low pass filter with normalized cutoff frequency $F = f_c / f_s$, $F < 1/2$. The inverse Fourier transform of this function is

$$x(t) = \int_{-F}^{F} e^{2\pi\, j\, f\, t} df = \frac{1}{2\pi\, j\, t}\left(e^{2\pi\, j\, F\, t} - e^{-2\pi\, j\, F\, t}\right) = \frac{1}{\pi\, t} sin(2\pi\, F\, t) \qquad (12.34)$$

Fact: The function sin(*t*) / (*t*) is the function sinc(*t*). Thus, finite impulse response filters are occasionally called *sinc functions*.

This function has ever increasing oscillations until $t = 0$, at which point it is $x(0) = 2\,F$, after which it has ever decreasing oscillations. To create a discrete time filter, we sample this continuous time function between -1 and 1 at the sampling frequency. We pick a discrete filter of length *N* from the middle *N* points of the sampled $x(t)$ and shift it to the right by $(N - 1) / 2$. This would produce the low pass filter below, which we also described in chapter 7 as the alternative form to our original low pass filter formula. This is, in fact, the more widely used formula and the more precise one.

The following is the formula for a finite impulse response low pass filter of length *N*, cutoff frequency f_c, given the sampling frequency f_s, and $k = 0, 1, \ldots, N - 1$. The usual transformations produce high pass, band pass, and band stop filters.

$$a(k) = \begin{cases} \dfrac{sin\left(\dfrac{2\pi\, f_c\left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\left(k - \frac{N-1}{2}\right)}, & k \neq \dfrac{N-1}{2} \\ 2\,\dfrac{f_c}{f_s}, & k = \dfrac{N-1}{2} \end{cases} \qquad (12.35)$$

A high pass filter can be produced by subtracting the coefficient of the low pass filter form those of an all pass filter.

$$a(k) = \begin{cases} -\dfrac{sin\left(\dfrac{2\pi\, f_c\, \left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\, \left(k - \frac{N-1}{2}\right)}, & k \neq \dfrac{N-1}{2} \\ 1 - 2\,\dfrac{f_c}{f_s}, & k = \dfrac{N-1}{2} \end{cases} \tag{12.36}$$

A band pass filter is the difference between two low pass filters at cutoff frequencies f_c and f_c', $f_c' < f_c$.

$$a(k) = \begin{cases} \dfrac{sin\left(\dfrac{2\pi\, f_c\, \left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\, \left(k - \frac{N-1}{2}\right)} - \dfrac{sin\left(\dfrac{2\pi\, f_c'\, \left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\, \left(k - \frac{N-1}{2}\right)}, & k \neq \dfrac{N-1}{2} \\ 2\,\dfrac{f_c}{f_s} - 2\,\dfrac{f_c'}{f_s}, & k = \dfrac{N-1}{2} \end{cases} \tag{12.37}$$

A band stop filter could be the difference between an all pass filter and a band pass filter. If $f_c' < f_c$

$$a(k) = \begin{cases} -\dfrac{sin\left(\dfrac{2\pi\, f_c\, \left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\, \left(k - \frac{N-1}{2}\right)} + \dfrac{sin\left(\dfrac{2\pi\, f_c'\, \left(k - \frac{N-1}{2}\right)}{f_s}\right)}{\pi\, \left(k - \frac{N-1}{2}\right)}, & k \neq \dfrac{N-1}{2} \\ 1 - 2\,\dfrac{f_c}{f_s} + 2\,\dfrac{f_c'}{f_s}, & k = \dfrac{N-1}{2} \end{cases} \tag{12.38}$$

12.6. Deriving the family of Hamming windows

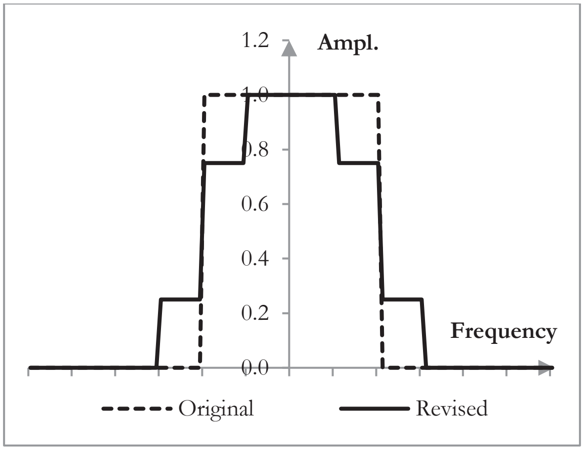
One usual purpose of windows is to mitigate the ripples of the Gibbs phenomenon, which are the result of the Fourier series approximation – a series of continuous functions – of the discontinuous desired magnitude response. One obvious thing to do to reduce these ripples is to use a different desired magnitude response, one in which this discontinuity is not as blatant. Take a look at figure 56. Rather than using the inverse Fourier transform on the ideal magnitude response (dashed), we will do so on a revised magnitude response (solid).

We can recognize that the revised magnitude response is the sum of the scaled original magnitude response $H(f)$ with two scaled and shifted versions of itself.

$$\alpha\, H(f) + \beta\, H(f - f_0) + \beta\, H(f + f_0) \tag{12.39}$$

where α is the scaling factor for the original magnitude response, β is the scaling factor for the shifted magnitude responses, and f_0 and $-f_0$ are the two shifts. In the example above, α = 0.5, β = 0.25, and $f_0 = ¼\, f_c$, where f_c is the cutoff frequency.

Figure 56. A revised desired magnitude response



The discontinuities of the revised desired magnitude response are softer.

We can also note that the windowed filter is the product of the filter and the window. According to the *convolution theorem*, the Fourier transform of the product of two functions is the convolution of the Fourier transforms of the two functions – this is a theorem that will help us explain a number of concepts in this book. If we are to use the inverse Fourier transform on some new (presumably, windowed) magnitude response, then this desired magnitude response must be a convolution of the Fourier transform of the filter and the Fourier transform of the window. We rewrite the revised magnitude response above as a convolution.

$$H(f) * \big(\alpha\, \delta(f) + \beta\, \delta(f - f_0) + \beta\, \delta(f + f_0)\big) \tag{12.40}$$

where δ is the Dirac delta function. We use the Dirac delta function, as it allows us to both write the desired magnitude response as a convolution, as well as to recognize the transform of the cosine function in the last two terms. The inverse Fourier transform of this desired magnitude response is the filter before the window, multiplied by the window

$$w(k) = \alpha + 2\,\beta\ cos(f_0\,k) \tag{12.41}$$

This is the generalized Hamming window family, which includes the Hamming window itself and the Hann window.

12.7. Dolph-Chebychev window

We will put the inverse Fourier transform to more practical use with the Dolph-Chebychev window. The Dolph-Chebychev windows are interesting, because they control the ripples in the pass and stop bands.

To cut down on notation, we introduce angular frequencies. Angular frequencies, usually denoted with ω, measure the number of rotations per unit of time. They are typically computed in radians per second, where radians measure the ratio of an arc of a circle to the radius of a circle. Since the arc of one rotation is the full circle, or 2π radians, a frequency of one rotation per second is the frequency of 2π radians per second. Thus, we can also say that if something has the frequency f, it also has the angular frequency $\omega = 2\pi f$. For example, $f = 40$ Hz means $\omega = 251.3$ radians per second.

The frequency f is also the angular frequency

$$\omega = 2\pi\,f \tag{12.42}$$

f is normalized, when divided by the sampling frequency f_s. By the Nyquist-Shannon sampling theorem, normalized frequencies are between 0 and ½, or between -½ and ½, if negative frequencies are allowed. Thus, normalized angular frequencies span the interval between $-\pi$ and π.

When angular frequencies are normalized, the sampling frequency itself normalizes to $\omega_s = 2\pi$. Suppose that we have the angular frequency $\omega = 1$ and the signal is sampled at $f_s = 2000$ Hz. Then in reality the frequency ω is equivalent to $1 * 2000 / 2\pi = 318.3$ Hz.

To design the Dolph-Chebychev window, we begin with the Dolph-Chebychev desired magnitude response.

$$|H(\omega)| = \frac{T_{2M}\left(\frac{cos\left(\frac{\omega}{2}\right)}{cos\left(\frac{\omega_0}{2}\right)}\right)}{T_{2M}\left(\frac{1}{cos\left(\frac{\omega_0}{2}\right)}\right)} \tag{12.43}$$

Here T_n are the Chebychev polynomials

$$T_n(x) = \begin{cases} cos(n\ acos(x)), & |x| \leq 1 \\ cosh(n\ acosh(x)), & |x| > 1 \end{cases} \tag{12.44}$$

M is an integer parameter, typically set to $M = (N - 1) / 2$, where N is the length of the desired filter, and cosh and acosh are the hyperbolic cosine and arccosine

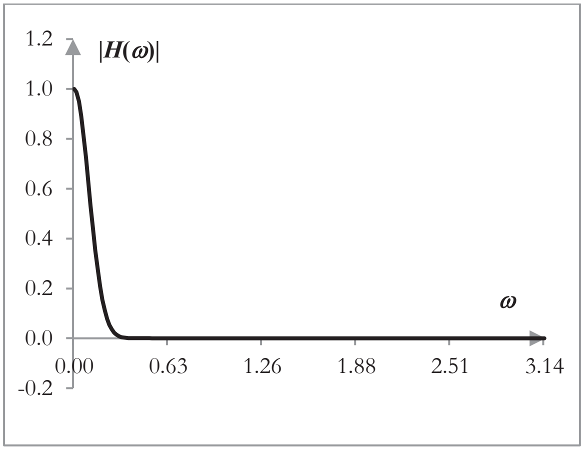
$$cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right) \tag{12.45}$$

$$acosh(x) = ln\left(x + \sqrt{x^2 - 1}\right),\ \ x \geq 1 \tag{12.46}$$

As ω increases between 0 and ω_0, $|H(\omega)|$ decreases from 1 to $1 / T_{2M}\ (1 / \cos(\omega_0 / 2))$. As ω continues between ω_0 and π, $|H(\omega)|$ oscillates between $1 / T_{2M}\ (1 / \cos(\omega_0 / 2))$ and $-1 / T_{2M}\ (1 / \cos(\omega_0 / 2))$. An example of $|H(\omega)|$ for $\omega_0 = 1$ and $N = 201$ is shown on figure 57.

This function could be a desired magnitude response of a low pass filter, although it decreases very quickly. Even though this is difficult to see on the graph, ω_0 is the frequency at which $H(\omega)$ reaches its "stop band." The oscillations in the stop band are extremely small. At $\omega_0 = 1$ and $M = 100$, $1 / T_{2M}\ (1 / \cos(\omega_0 / 2)) = 8.71\text{E-}46$.

Figure 57. The Dolph-Chebychev magnitude response



The Dolph-Chebychev magnitude response with M = 100 and ω_0 = 1.

We can compute a low pass filter $a(k)$ – the *Dolph-Chebychev filter* – from $|H(\omega)|$ by taking its generalized inverse DFT as above. In closed form, the filter would be as follows.

$$a(k) = \frac{1}{N}\sum_{n=0}^{M} \frac{T_{2M}\left(\frac{cos(\omega_n/2)}{cos(\omega_0/2)}\right)}{T_{2M}\left(\frac{1}{cos(\omega_0/2)}\right)} cos\big((k-M)\omega_n\big) \tag{12.47}$$

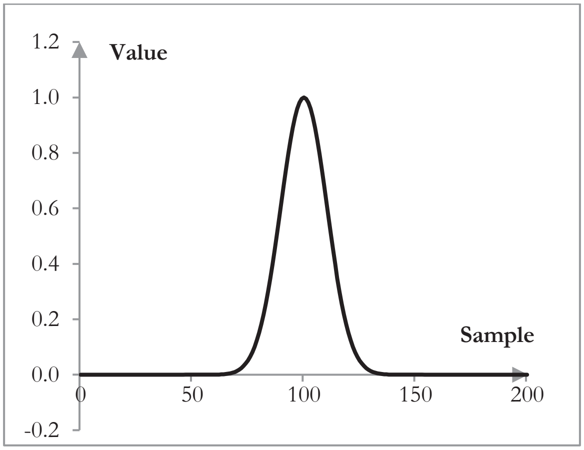
where k = 0, 1, …, N – 1, N is the length of the filter, M = (N – 1) / 2, and ω_n are the frequencies of the N inverse DFT components, $\omega_n = n\,\pi$ / (N – 1).

The *Dolph-Chebychev window* $w(k)$ is simply the Dolph-Chebychev filter scaled to ensure that it peaks at 1.

$$w(k) = \frac{a(k)}{a(M)} \tag{12.48}$$

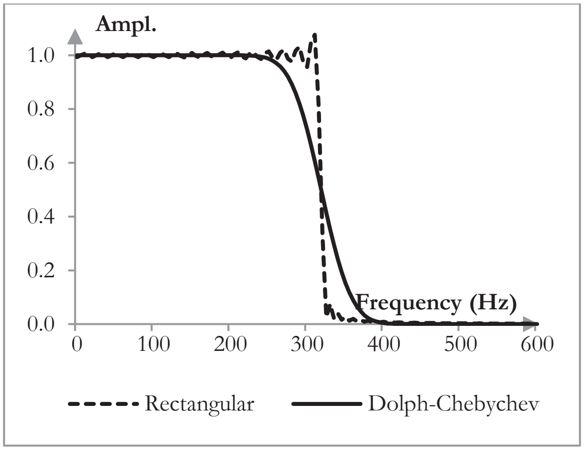
Suppose that N = 201, M = 100, and ω_0 = 1. The resulting window is shown on figure 58. Suppose that the sampling frequency is 2000 Hz. Take a low pass filter with a cutoff frequency of 318.3 Hz (i.e., ω_0 = 318.3 * 2π / 2000 = 1). The magnitude response of the resulting filter is shown on figure 59.

Figure 58. Dolph-Chebychev window



A Dolph-Chebychev window with $N = 201$, $M = 100$, and $\omega_0 = 1$. The Dolph-Chebychev window depresses the far ends of the lobes of the original impulse response.

Figure 59. Magnitude response of a filter with the Dolph-Chebychev window



Magnitude response (normalized amplitude) of a low pass filter with the Dolph-Chebychev window ($N = 201$, $M = 100$, and $\omega_0 = 1$).

Note that, although the Dolph-Chebychev filter has the known stop band attenuation $1 / T_{N-1}(1 / \cos(\omega_0 / 2))$, a filter with the Dolph-Chebychev window can have different stop band attenuation, which will depend on the filter's cutoff frequency and length.

While we can think of the Dolph-Chebychev window as the "Dolph-Chebychev filter" – the inverse Fourier transform of the Dolph-Chebychev magnitude response function, this is not the most intuitive motivation for the window. It is better to note that the Dolph-Chebychev magnitude response function in figure 57 resembles an impulse. Note also that the Fourier transform of the product of two functions – the filter and the window – is the convolution of two Fourier transforms – the transform of the filter and the transform of the window.

$$\mathcal{F}(f\ w) = \mathcal{F}(f) * \mathcal{F}(w) \tag{12.49}$$

In other words, when a window is applied to a filter, the magnitude response of the filter before the window and the magnitude response of the window are convolved. The magnitude response of the window acts as a filter on the magnitude response of the filter before the window.

The smaller the value of ω_0 in the Dolph-Chebychev window, the more the Dolph-Chebychev magnitude response function (the Fourier transform of the window itself) resembles an impulse and acts similarly to an all pass filter on the magnitude response of the filter. The magnitude response of the filter remains almost unchanged, as if applying a rectangular window. With smaller ω_0, the Dolph-Chebychev window is closer to a rectangular window. With larger ω_0, the lobe in the magnitude response graph above becomes larger, and the Dolph-Chebychev window creates a smoother magnitude response for the filter.

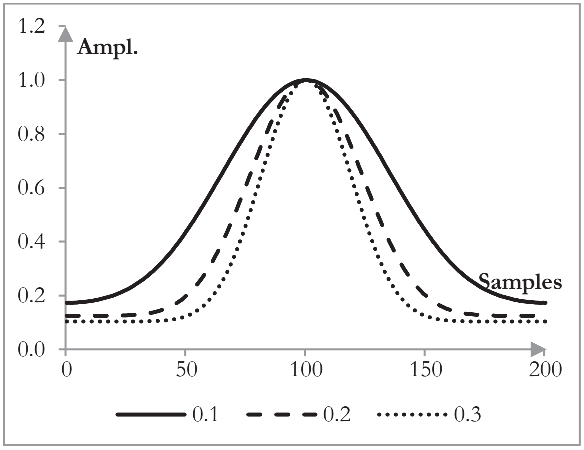
Although we could think of ω_0 as a "cutoff frequency," we should not. It is simply a constant parameter. The values of ω_0 are usually small (< 0.1; although the examples provided below use larger values due to issues with precision). Other definitions of the Dolph-Chebychev window may use

$$\frac{1}{cos(\omega_0/2)} = cosh\left(\frac{1}{N}\ acosh(10^{\alpha})\right) \tag{12.50}$$

Here, the constant parameter $\alpha > 0$ ensures that $1\ /\ \cos(\omega_0\ /\ 2) \geq 1$ and that $1\ /\ \cos(\omega_0\ /\ 2)$ is approximately equal to 1. This in turn creates a Dolph-Chebychev window, the main lobe (of the window, not the filter) of which is relatively wide and the side portions of the window – the ones that compress the side lobes of the filter – are relatively short. This is equivalent to using values for ω_0 that are close to zero.

Figure 60 shows the Dolph-Chebychev window for three different values of ω_0 (0.1, 0.2, and 0.3), with $N = 201$ and $M = 100$.

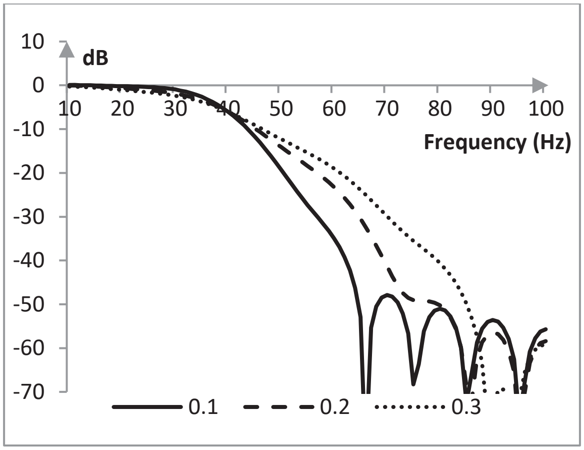
Figure 60. Dolph-Chebychev windows at three different ω_0



As ω_0 increases, the main lobe of the window becomes narrower, and the flat side portions become larger.

The magnitude responses of the three corresponding filters, given a sampling rate of 2000 Hz and a cutoff frequency of 40 Hz, are shown on figure 61.

Figure 61. Magnitude response of low pass filters with Dolph-Chebychev windows with different ω_0



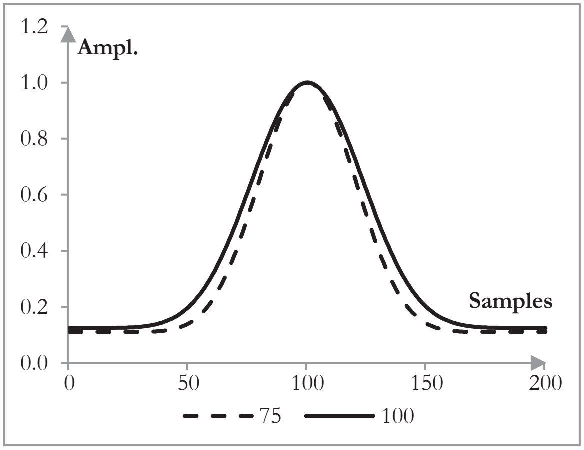
As ω_0 increases, the stop band attenuation of the filter with the Dolph-Chebychev window improves, but its transition band becomes larger.

In principle, when designing the Dolph-Chebychev window, there is no need to link the parameter M in the Chebychev polynomials T_{2M} to the number of points N in the inverse Fourier transform (in the formulae above $M = (N - 1) / 2$). We can write the formula for the Dolph-Chebychev window as

$$a_0(k) = \frac{1}{N}\sum\nolimits_{n=0}^{M} \frac{T_{2L}\left(\frac{cos(\omega_n/2)}{cos(\omega_0/2)}\right)}{T_{2L}\left(\frac{1}{cos(\omega_0/2)}\right)}\ cos((k-M)\ \omega_n) \qquad (12.51)$$

where there is no relationship between *N* and *L*. Figure 62 shows the Dolph-Chebychev window of the previous example with *ω*₀ = 0.2 and at *L* = 100 and *L* = 75.

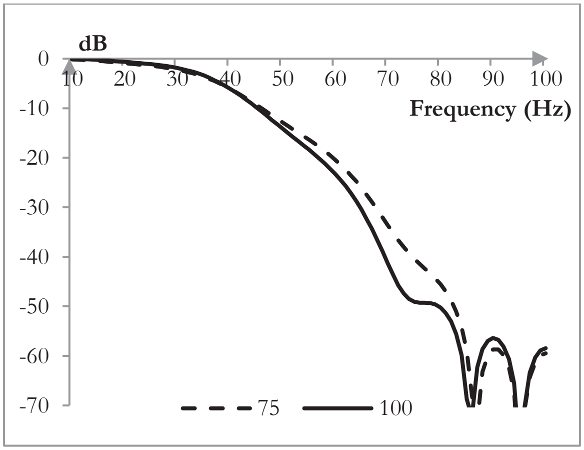
Figure 62. Dolph-Chebychev window with two different *L*



As *L* increases, the main lobe of the window becomes wider.

The magnitude responses of the corresponding filters of length 201 and cutoff frequency 40 Hz are shown on figure 63.

Figure 63. Magnitude responses of low pass filters with the Dolph-Chebychev window at different *L*



All else equal, a larger *L* will produce a Dolph-Chebychev window with a wider lobe and a corresponding filter with a better stop band attenuation, but a wider transition band.